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APPROXIMATION OF SOLUTIONS OF SDE's WITH OBLIQUE REFLECTION ON AN ORTHANT*

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Abstract. We consider the discrete penalization scheme, the projection and the Euler–Peano scheme for SDE's driven by general semimartingale on an orthant with oblique reflection. We prove that these schemes converge in probability to the solution of the SDE in various topologies provided that the oblique reflection satisfies the assumption introduced by Harrison and Reiman. In the case where the driving semimartingale is an Itô process, the rate of L^p-convergence is discussed in detail.

Mathematics Subject Classification: Primary 60H20; Secondary 60H99.

1. INTRODUCTION

Suppose we are given a d-dimensional semimartingale $Z = (Z^1, \ldots, Z^d)$, a Lipschitz continuous function $\sigma\colon \boldsymbol{R}^d \to \boldsymbol{R}^d \otimes \boldsymbol{R}^d$, and a nonnegative $d \times d$ matrix Q with zeros on the diagonal and spectral radius $\varrho(Q)$ strictly less than 1. Consider a d-dimensional stochastic differential equation (SDE) on an orthant $(\boldsymbol{R}^+)^d$ with oblique reflection of the form

$$X_t = X_0 + \int_0^t \sigma(X_{s-})\,dZ_s + (I-Q')K_t, \quad t \in \boldsymbol{R}^+. \tag{1.1}$$

Here Q' is the transpose of Q, $X_0 \in (\boldsymbol{R}^+)^d$, $X = (X^1, \ldots, X^d)$ is a reflecting process on $(\boldsymbol{R}^+)^d$, and $K = (K^1, \ldots, K^d)$ is a process with nondecreasing trajectories such that K^j increases only at those times t where $X^j_t = 0$. Equations of type (1.1) were introduced in the paper by Harrison and Reiman [8] in the case $\sigma = I$, $Z = W$, where W is a d-dimensional standard Wiener process, and considered later by Yamada [30] in the case of reflecting Itô diffusions. SDE's driven by continuous semimartingale under milder assumptions on Q were

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discussed in Dupuis and Ishi [4], [5] (see also [1], [6], [7], [9], [28], [29] for some related results).

The main purpose of the present paper is to investigate three numerical methods of approximation of the solution X to the SDE (1.1): the discrete penalization scheme, the projection scheme, and the Euler–Peano scheme.

Let $\varrho_t^n = \max\{k/n;\ k\in N\cup\{0\},\ k/n \leqslant t\}$ and let $Z_t^{\varrho^n}$ be a discretization of Z, i.e. $Z_t^{\varrho^n} = Z_{k/n}$ for $t\in[k/n,\ (k+1)/n)$, $k\in N\cup\{0\}$, $n\in N$. In the first method the approximation processes $\hat{X}^n$ are defined to be the solutions of discrete penalized SDE's

$$\hat{X}_t^n = X_0+\int_0^t \sigma(\hat{X}_{s-}^n)\,dZ_s^{\varrho^n}-n\int_0^t (\hat{X}_{s-}^n-\Pi_Q(\hat{X}_{s-}^n))\,d\varrho_s^n,\quad t\in \boldsymbol{R}^+, \tag{1.2}$$

where $\Pi_Q(z)$ denotes an oblique projection of z on $(\boldsymbol{R}^+)^d$. In the second and third methods the approximation processes $\bar{X}^n$ and X^n are solutions of SDE's on $(\boldsymbol{R}^+)^d$ with oblique reflection of the forms

$$\bar{X}_t^n = X_0+\int_0^t \sigma(\bar{X}_{s-}^n)\,dZ_s^{\varrho^n}+(I-Q')\bar{K}_t^n,\qquad t\in \boldsymbol{R}^+, \tag{1.3}$$

and

$$X_t^n = X_0+\int_0^t \sigma(X_{s-}^{n,\varrho^n})\,dZ_s+(I-Q')K_t^n,\qquad t\in \boldsymbol{R}^+, \tag{1.4}$$

respectively (see Sections 2 and 3 for precise definitions). Note that (1.2) is a counterpart to a discrete penalization scheme introduced in Liu [16] in the case of the Itô SDE with normal reflection (see also Pettersson [20], Słomiński [26]), whereas (1.3) and (1.4) are counterparts to the well-known projection scheme and the Euler–Peano scheme considered earlier in Chitashvili and Lazrieva [2], Lépingle [15], Pettersson [19], [20] and Słomiński [24], [26].

We will see in Section 3 that $\hat{X}^n$, $\bar{X}^n$ and X^n can be computed by simple reccurent formulas. Moreover, we prove that $\{\hat{X}^n\}$, $\{\bar{X}^n\}$ and $\{X^n\}$ converge in probability to X in the space $\boldsymbol{D}(\boldsymbol{R}^+, \boldsymbol{R}^d)$ endowed with the S-topology introduced recently by Jakubowski [10], the Skorokhod J_1-topology and the uniform topology, respectively (of course, J_1 is weaker than the uniform topology, and S is weaker than J_1). Finally, in the case of the Euler–Peano scheme we give some results on the rate of uniform convergence as well as convergence in variation.

In Section 4 we deal with Itô diffusions reflecting on an orthant, i.e. with solutions to the SDE

$$X_t = X_0+\int_0^t \sigma(X_s)\,dW_s+\int_0^t b(X_s)\,ds+(I-Q')K_t,\qquad t\in \boldsymbol{R}^+, \tag{1.5}$$

where $X_0 \in \bar{D}$. We prove that if σ, b are Lipschitz continuous, then for every $p \in N$

$$E \sup_{s \leqslant t} |\hat{X}_s^n - X_s|^{2p} = \mathcal{O}\left(((\ln n)/n)^p\right), \quad t \in \boldsymbol{R}^+,$$

and

$$E \sup_{s \leqslant t} |\bar{X}_s^n - X_s|^{2p} = \mathcal{O}\left(((\ln n)/n)^p\right), \quad t \in \boldsymbol{R}^+,$$

whereas in the case of the Euler–Peano scheme

$$E \sup_{s \leqslant t} |X_t^n - X_t|^{2p} = \mathcal{O}\left((n^{-1})^p\right), \quad t \in \boldsymbol{R}^+.$$

In the case important in the applications, where the diffusion coefficient is constant, we are able to estimate the total variation $|X^n - X|_t$ of $X^n - X$ on the interval $[0, t]$. Namely, we prove that

$$E |X^n - X|_t^p = \mathcal{O}\left((n^{-1})^{p/2}\right), \quad t \in \boldsymbol{R}^+.$$

For convenience of the reader we collect in the Appendix some basic results concerning the condition (UT) and the estimates of the solutions of SDE's.

Throughout the paper $\rightarrow_{\mathcal{P}}$ denotes convergence in probability and $\boldsymbol{D}(\boldsymbol{R}^+, \boldsymbol{R}^d)$ denotes the space of functions $x\colon \boldsymbol{R}^+ \to \boldsymbol{R}^d$ which are right continuous and admit left limits. For given $y \in \boldsymbol{D}(\boldsymbol{R}^+, \boldsymbol{R}^d)$ we write $\Delta y_t = y_t - y_{t-}$ and we denote by $\omega_h(y, t)$ the modulus of continuity of y on $[0, t]$, i.e.

$$\omega_h(y, t) = \sup_{\{u,v \in [0,t], |u-v| \leqslant h\}} |y_u - y_v|, \quad h > 0, \ t \in \boldsymbol{R}^+.$$

2. THE SKOROKHOD PROBLEM ON AN ORTHANT

Let Q be a nonnegative matrix with zeros on the diagonal and $\varrho(Q) < 1$ and let $y \in \boldsymbol{D}(\boldsymbol{R}^+, \boldsymbol{R}^d)$ with $y_0 \in (\boldsymbol{R}^+)^d$. Following Harrison and Reiman [8] a pair $(x, k) \in \boldsymbol{D}(\boldsymbol{R}^+, \boldsymbol{R}^{2d})$ is called a *solution to the Skorokhod problem*

(2.1) $$x_t = y_t + (I - Q')\, k_t, \quad t \in \boldsymbol{R}^+,$$

on $(\boldsymbol{R}^+)^d$ associated with y if (2.1) is satisfied and

(2.2) $$x_t \in (\boldsymbol{R}^+)^d, \quad t \in \boldsymbol{R}^+,$$

(2.3) k^j is nondecreasing, $k_0^j = 0$

$$\text{and} \quad \int_0^t x_s^j\, dk_s^j = 0 \quad \text{for } j = 1, \ldots, d, \ t \in \boldsymbol{R}^+.$$

Remark 2.1 ([8], Theorem 1). (i) For every $y \in \boldsymbol{D}(\boldsymbol{R}^+, \boldsymbol{R}^d)$ with $y_0 \in (\boldsymbol{R}^+)^d$ there exists a unique solution (x, k) of the Skorokhod problem associated with y. Moreover, if (x', k') denotes a solution of the Skorokhod problem associated with $y' \in \boldsymbol{D}(\boldsymbol{R}^+, \boldsymbol{R}^d)$ such that $y'_0 \in (\boldsymbol{R}^+)^d$, then there exists a constant C depending only on the matrix Q such that

$$(2.4)\qquad \sup_{s \leqslant t} |x'_s - x_s| \leqslant C \sup_{s \leqslant t} |y'_s - y_s|, \quad t \in \boldsymbol{R}^+.$$

(ii) If additionally $\|Q\| < 1$, where $\|A\|$ denotes the maximum row sum of a matrix A, then k is the unique fixed point of the mapping $F\colon \boldsymbol{D}(\boldsymbol{R}^+, \boldsymbol{R}^d) \to \boldsymbol{D}(\boldsymbol{R}^+, \boldsymbol{R}^d)$ defined by the formula

$$(2.5)\qquad F(u)_t = \sup_{s \leqslant t} [Q' u_s - y_s]^+, \quad u \in \boldsymbol{D}(\boldsymbol{R}^+, \boldsymbol{R}^d),\ t \in \boldsymbol{R}^+,$$

where $[z]^+ = \max(0, z)$. In this case a constant C in (2.4) takes the form $C = 1 + C_Q$, where $C_Q = 1/(1 - \|Q\|)$.

Suppose that

$$(2.6)\qquad \|Q\| < 1$$

and for given $z \in \boldsymbol{R}^d$ consider the mapping $f\colon \boldsymbol{R}^d \to \boldsymbol{R}^d$, $f(v) = [Q' v - z]^+$, $v \in \boldsymbol{R}^d$. Since $|f(v) - f(v')| \leqslant \|Q\| \, |v - v'|$ for $v, v' \in \boldsymbol{R}^d$, it follows by (2.6) that f is Lipschitz continuous with a constant less than 1, and hence that for every $z \in \boldsymbol{R}^d$ there is the unique fixed point $\bar{z}$ of f. Therefore, under (2.6), we may define a mapping $\Pi_Q\colon \boldsymbol{R}^d \to (\boldsymbol{R}^+)^d$ by

$$(2.7)\qquad \Pi_Q(z) = (I - Q')\bar{z} + z, \quad z \in \boldsymbol{R}^d,$$

which is called an *oblique projection* on an orthant $(\boldsymbol{R}^+)^d$.

Suppose now that (2.6) does not longer hold. Since $\varrho(Q) < 1$, there exists a diagonal matrix Λ having positive elements and a nonnegative matrix Q^* such that $\|Q^*\| < 1$ and $Q' = \Lambda^{-1} \circ (Q^*)' \circ \Lambda$. Define Π_{Q^*} by (2.7) and set

$$(2.8)\qquad \Pi_Q(z) = \Lambda^{-1} \circ \Pi_{Q^*} \circ \Lambda(z), \quad z \in \boldsymbol{R}^d.$$

The matrices Λ and Q^* are not uniquely determined by Q. We have, however, the following

LEMMA 2.2. *The function Π_Q defined by* (2.8) *does not depend on the choice of Λ and Q^*. Moreover, $\Pi_Q\colon \boldsymbol{R}^d \to (\boldsymbol{R}^+)^d$ is a Lipschitz continuous mapping with a constant depending only on the matrix Q.*

Proof. Note first that (x, k) is a solution of the Skorokhod problem associated with y if and only if $(\tilde{x} = \Lambda \circ x, \tilde{k} = \Lambda \circ k)$ is a solution of the Skorokhod problem associated with $\tilde{y} = \Lambda \circ y$. Fix $z \in \boldsymbol{R}^d$ and set

$$y_t = \begin{cases} 0 & \text{if } t < 1, \\ z & \text{otherwise.} \end{cases}$$

It is clear that $(\tilde{x}, \tilde{k})$, where

$$\tilde{x}_t = \begin{cases} 0 & \text{if } t < 1, \\ \Pi_{Q^*} \circ \Lambda(z) & \text{otherwise,} \end{cases}$$

and $\tilde{k}_t = \tilde{x}_t - \Lambda \circ y_t$, $t \in \boldsymbol{R}^+$, is the unique solution of the Skorokhod problem associated with $\tilde{y} = \Lambda \circ y$. Hence the unique solution of the Skorokhod problem associated with y equals (x, k), where

$$x_t = \Lambda^{-1} \circ \tilde{x}_t = \begin{cases} 0 & \text{if } t < 1, \\ \Lambda^{-1} \circ \Pi_{Q^*} \circ \Lambda(z) & \text{otherwise,} \end{cases}$$

and $k_t = x_t - y_t$, $t \in \boldsymbol{R}^+$. Hence, by uniqueness of the solution of the Skorokhod problem, the value of $\Lambda^{-1} \circ \Pi_{Q^*} \circ \Lambda(z)$ does not depend on Λ and Q^*. To prove the Lipschitz continuity, for fixed $z' \in \boldsymbol{R}^d$ set

$$y'_t = \begin{cases} 0 & \text{if } t < 1, \\ z' & \text{otherwise.} \end{cases}$$

Then, as above, the pair (x', k'), where

$$x'_t = \begin{cases} 0 & \text{if } t < 1, \\ \Pi_Q(z') & \text{otherwise,} \end{cases}$$

$k'_t = x'_t - y'_t$, $t \in \boldsymbol{R}^+$, is the unique solution of the Skorokhod problem associated with y'. By (2.4),

$$|\Pi_Q(z) - \Pi_Q(z')| = \sup_{s \leqslant 1} |x'_s - x_s| \leqslant C \sup_{s \leqslant 1} |y'_s - y_s| = |z - z'|,$$

which completes the proof. ∎

In view of Lemma 2.2, (2.8) defines correctly an oblique projection for any Q with $\varrho(Q) < 1$.

EXAMPLE 2.3. In the case $d = 2$ the mapping Π_Q is given by the following simple formulas:

$$\Pi_Q\big(z = (z_1, z_2)\big) = \begin{cases} (z_1, z_2) & \text{if } z_1, z_2 \geqslant 0, \\ (z_1 + Q_{21} z_2, 0) & \text{if } z_1 + Q_{21} z_2 \geqslant 0,\ z_2 < 0, \\ (0, Q_{12} z_1 + z_2) & \text{if } z_1 < 0,\ Q_{12} z_1 + z_2 \geqslant 0, \\ (0, 0) & \text{if } z_1 + Q_{21} z_2 < 0,\ z_1 + Q_{21} z_2 < 0. \end{cases}$$

Let (x, k) be a solution of the Skorokhod problem associated with $y \in \boldsymbol{D}(\boldsymbol{R}^+, \boldsymbol{R}^d)$. By using the oblique projection mapping Π_Q we define recurrently for each $n \in \boldsymbol{N}$ the approximations $(\hat{x}^n, \hat{k}^n)$ and $(\bar{x}^n, \bar{k}^n)$ of (x, k) by the formulas

$$\hat{x}^n_0 = \bar{x}^n_0 = y_0,$$

$$\hat{x}^n_{(k+1)/n} = \Pi_Q(\hat{x}^n_{k/n}) + (y_{(k+1)/n} - y_{k/n}), \qquad \bar{x}^n_{(k+1)/n} = \Pi_Q\big(\bar{x}^n_{k/n} + (y_{(k+1)/n} - y_{k/n})\big),$$

$$\hat{x}^n_t = \hat{x}^n_{k/n}, \qquad \bar{x}^n_t = \bar{x}^n_{k/n}, \qquad t \in [k/n, (k+1)/n)$$

for $k \in N \cup \{0\}$ and

$$\hat{k}_t^n = (I-Q')^{-1}(\hat{x}_t^n - y_t^{\varrho^n}), \quad \bar{k}_t^n = (I-Q')^{-1}(\bar{x}_t^n - y_t^{\varrho^n}), \quad t \in [k/n, (k+1)/n),$$

where $y_t^{\varrho^n} = y_{k/n}$, $k \in N \cup \{0\}$.

THEOREM 2.4. *If* $y \in \boldsymbol{D}(\boldsymbol{R}^+, \boldsymbol{R}^d)$, $y_0 \in (\boldsymbol{R}^+)^d$, *then*

$$(\hat{x}^n, \hat{k}^n) \to (x, k) \quad \text{in } (\boldsymbol{D}(\boldsymbol{R}^+, \boldsymbol{R}^{2d}), S) \tag{2.9}$$

and

$$(\bar{x}^n, \bar{k}^n) \to (x, k) \quad \text{in } (\boldsymbol{D}(\boldsymbol{R}^+, \boldsymbol{R}^{2d}), J_1). \tag{2.10}$$

Proof. We start with the proof of (2.10). Without loss of generality we may and will assume (2.6). First observe that $(\bar{x}^n, \bar{k}^n)$ is a solution of the Skorokhod problem associated with y^{ϱ^n} and $y^{\varrho^n} \to y$ in $(\boldsymbol{D}(\boldsymbol{R}^+, \boldsymbol{R}^d), J_1)$, i.e. there exists a sequence $\{\lambda^n\}$ of strictly increasing continuous changes of time such that $\lambda_0^n = 0$, $\lambda_\infty^n = +\infty$ and

$$\sup_{s \leqslant t} |\lambda_s^n - s| \to 0, \quad \sup_{s \leqslant t} |y_{\lambda_s^n}^{\varrho^n} - y_s| \to 0 \quad \text{for } t \in \boldsymbol{R}^+.$$

Since $\bar{k}_{\lambda_t^n}^n = \sup_{s \leqslant t} [Q' \bar{k}_{\lambda_s^n}^n - y_{\lambda_s^n}^{\varrho^n}]^+$ for $n \in N$, $t \in \boldsymbol{R}^+$, we have

$$\sup_{s \leqslant t} |\bar{k}_{\lambda_s^n}^n - k_s| \leqslant \|Q\| \sup_{s \leqslant t} |\bar{k}_{\lambda_s^n}^n - k_s| + \sup_{s \leqslant t} |y_{\lambda_s^n}^{\varrho^n} - y_s|.$$

By the above and (2.6) we have $\sup_{s \leqslant t} |\bar{k}_{\lambda_s^n}^n - k_s| \to 0$, $\sup_{s \leqslant t} |\bar{x}_{\lambda_s^n}^n - x_s| \to 0$, which implies (2.10). Now, write

$$k_t^n = \sum_{k;(k+1)/n \leqslant t} \left(\Pi_Q(\hat{x}_{(k+1)/n}^n) - \hat{x}_{(k+1)/n}^n\right), \quad t \in [k/n, (k+1)/n),\ k \in N \cup \{0\},\ n \in N,$$

and observe that $(\Pi_Q(\hat{x}^n), k^n)$ is a solution of the Skorokhod problem associated with y^{ϱ^n}. Hence $(\Pi_Q(\hat{x}^n), k^n) = (\bar{x}^n, \bar{k}^n)$, by uniqueness. On the other hand,

$$|\Pi_Q(\hat{x}_{(k+1)/n}^n) - \hat{x}_{(k+1)/n}^n| = |\Delta \bar{k}_{(k+1)/n}^n|$$
$$= [Q' \Delta \bar{k}_{(k+1)/n}^n - \Delta y_{(k+1)/n}^{\varrho^n} - \bar{x}_{k/n}^n]^+ \leqslant \|Q\| \, |\Delta \bar{k}_{(k+1)/n}^n| + |\Delta y_{(k+1)/n}^{\varrho^n}|,$$

which implies that, for every $k \in N \cup \{0\}$,

$$|\Pi_Q(\hat{x}_{(k+1)/n}^n) - \hat{x}_{(k+1)/n}^n| \leqslant C_Q |\Delta y_{(k+1)/n}^{\varrho^n}|. \tag{2.11}$$

By (2.10) and (2.11), $(\hat{x}_t^n, \hat{k}_t^n) \to (x_t, k_t)$ for every continuity point t of y, and, moreover, $\sup_n \sup_{s \leqslant t} |\hat{x}_s^n| < +\infty$, $\sup_n \sup_{s \leqslant t} |\hat{k}_s^n| < +\infty$ for $t \in \boldsymbol{R}^+$. Thus, in view of [10], Proposition 2.14, in order to complete the proof it suffices to prove that the sequences $\{\hat{x}^n\}$ and $\{\hat{k}^n\}$ are relatively S-compact. Since $\hat{x}^n = y^{\varrho^n} + \hat{k}^n$ and $\{y^{\varrho^n}\}$ is relatively S-compact, what is left to show is that $\{\hat{x}^n\}$ is relatively S-compact or, by [10], Lemma 2.1, that for every $\varepsilon > 0$ there is a sequence $\{v^{n,\varepsilon}\} \subset \boldsymbol{D}(\boldsymbol{R}^+, \boldsymbol{R}^d)$ of functions with locally bounded variation such that

$$\sup_n \sup_{s\leqslant t} |\hat{x}^n_s - v^{n,\varepsilon}_s| \leqslant \varepsilon, \quad t\in \boldsymbol{R}^+, \tag{2.12}$$

$$\sup_n |v^{n,\varepsilon}|_t < +\infty, \quad t\in \boldsymbol{R}^+. \tag{2.13}$$

By (2.10) for every $\varepsilon > 0$ there exists a sequence $\{a^{n,\varepsilon}\} \subset \boldsymbol{D}(\boldsymbol{R}^+, \boldsymbol{R}^d)$ of functions with locally bounded variation such that

$$\sup_n \sup_{s\leqslant t} |\Pi_Q(\hat{x}^n)_s - a^{n,\varepsilon}_s| \leqslant \varepsilon/2, \quad \sup_n |a^{n,\varepsilon}|_t < +\infty \quad \text{for } t\in \bar{\boldsymbol{R}}^+.$$

Set $K(n, \varepsilon) = \{k;\ |\Delta y^{\varrho^n}_{(k+1)/n}| > \varepsilon/(2C_Q)\}$, $n\in \boldsymbol{N}$. By (2.11), for every $k\in K(n, \varepsilon)$ there is a matrix $C(n, k)$ such that

$$\max_{n,k} \|C(n, k)\| < \infty \quad \text{and} \quad \hat{x}^n_{(k+1)/n} - \Pi_Q(\hat{x}^n_{(k+1)/n}) = C(n, k)\,\Delta y^{\varrho^n}_{(k+1)/n}.$$

Put

$$b^{n,\varepsilon}_t = \sum_{k\in K(n,\varepsilon)} C(n, k)\,\Delta y^{\varrho^n}_{(k+1)/n} \mathbf{1}_{[(k+1)/n,(k+2)/n)}(t), \quad n\in \boldsymbol{N}.$$

Since $y\in \boldsymbol{D}(\boldsymbol{R}^+, \boldsymbol{R}^d)$, $\sup_n |b^{n,\varepsilon}|_t < +\infty$. Therefore, it is clear that $v^{n,\varepsilon}$ defined by $v^{n,\varepsilon}_t = a^{n,\varepsilon}_t + b^{n,\varepsilon}_t$, $t\in \boldsymbol{R}^+$, $n\in \boldsymbol{N}$, satisfies (2.12) and (2.13). ∎

COROLLARY 2.5. *Under the assumptions of Theorem* 2.4 *there exists a constant C depending only on Q such that*

$$\sup_{s\leqslant t} |\bar{x}^n_s - x_s| \leqslant C\omega_{1/n}(y, t), \quad t\in \boldsymbol{R}^+, \tag{2.14}$$

and

$$\sup_{s\leqslant t} |\hat{x}^n_s - x_s| \leqslant C\omega_{1/n}(y, t), \quad t\in \boldsymbol{R}^+. \tag{2.15}$$

Proof. Since $(\bar{x}^n, \bar{k}^n)$ is a solution of the Skorokhod problem associated with y^{ϱ^n}, from (2.4) we get

$$\sup_{s\leqslant t} |\bar{x}^n_s - x_s| \leqslant C \sup_{s\leqslant t} |y^{\varrho^n}_s - y_s| \leqslant C\omega_{1/n}(y, t), \quad t\in \boldsymbol{R}^+.$$

The assertion (2.15) follows from (2.11) and the fact that $\Pi_Q(\hat{x}^n) = \bar{x}^n$. ∎

EXAMPLE 2.6. The following example shows that (2.9) cannot be strengthened to the convergence in J_1 or M_1. Let $d = 1$, $Q = 0$ and

$$y_t = \begin{cases} 0 & \text{if } t < 1, \\ -1 & \text{otherwise.} \end{cases}$$

Then $x_t = 0$, $t \in \boldsymbol{R}^+$, and by simple calculations, $\hat{x}_t^n = \mathbb{1}_{[1+2/n, 1+3/n)}(t)$, $t \in \boldsymbol{R}^+$, $n \in N$, and so $\{\hat{x}^n\}$ does not converge to x either in $(\boldsymbol{D}(\boldsymbol{R}^+, \boldsymbol{R}^d), J_1)$ or in $(\boldsymbol{D}(\boldsymbol{R}^+, \boldsymbol{R}^d), M_1)$. Note, however, that $\hat{k}_t^n \to k_t$ for every continuity point of y. Since the functions k^n and k are nondecreasing, we conclude that $\hat{k}^n \to k$ in $(\boldsymbol{D}(\boldsymbol{R}^+, \boldsymbol{R}), M_1)$.

THEOREM 2.7. *Let $(x, k), (x', k')$ denote the solutions to the Skorokhod problem associated with $y, y' \in \boldsymbol{D}(\boldsymbol{R}^+, \boldsymbol{R}^d)$, respectively. If the total variation of $y - y'$ is locally bounded, then there is a constant C depending only on Q such that*

$$|x - x'|_t \leqslant C\,|y - y'|_t, \qquad t \in \boldsymbol{R}^+. \tag{2.16}$$

Proof. Without loss of generality we may and will assume that $\|Q\| < 1$. Let $\{(\bar{x}^n, \bar{k}^n)\}$ and $\{(\bar{x}'^{,n}, \bar{k}'^{,n})\}$ denote the sequences of solutions of the Skorokhod problem associated with $\{y^{\varrho^n}\}$ and $\{y'^{,\varrho^n}\}$, respectively. By [22], Lemma 1, and by an elementary inequality $|[a]^+ - [b]^+| \leqslant |a - b|$ we have

$$\sum_{k;(k+1)/n \leqslant t} |\Delta \bar{k}^n_{(k+1)/n} - \Delta \bar{k}'^{,n}_{(k+1)/n}|$$

$$\leqslant \|Q\| \sum_{k;(k+1)/n \leqslant t} |\Delta \bar{k}^n_{(k+1)/n} - \Delta \bar{k}'^{,n}_{(k+1)/n}| + \sum_{k;(k+1)/n \leqslant t} |\Delta y^{\varrho^n}_{(k+1)/n} - \Delta y'^{,\varrho^n}_{(k+1)/n}|,$$

which implies that $|\bar{k}^n - \bar{k}'^{,n}|_{\varrho_t^n} \leqslant C_Q\,|y^{\varrho^n} - y'^{,\varrho^n}|_{\varrho_t^n}$ with C_Q defined in Remark 2.1. Since $(y^{\varrho^n}, y'^{,\varrho^n}) \to (y, y')$ and, by (2.9), $(\bar{k}^n, k'^{,n}) \to (k, k')$ on $(\boldsymbol{D}(\boldsymbol{R}^+, \boldsymbol{R}^{2d}), J_1)$,

$$|k - k'|_t \leqslant \limsup_{n\to\infty} |\bar{k}^n - \bar{k}'^{,n}|_t \leqslant C_Q \limsup_{n\to\infty} |y^{\varrho^n} - y'^{,\varrho^n}|_t \leqslant C_Q\,|y - y'|_t$$

for every t such that $y_t = y_{t-}$ and $y'_t = y'_{t-}$. This gives (2.16), since y, y' are right continuous. ∎

EXAMPLE 2.8. Let $w^1, w^2 \in \boldsymbol{D}(\boldsymbol{R}^+, \boldsymbol{R}^d)$, $w_0^1 \in (\boldsymbol{R}^+)^d$. Define

$$y_t = w_t^1 + \int_0^t w_{s-}^2\, ds, \qquad y_t^n = w_t^1 + \int_0^t w_{s-}^{2,\varrho^n}\, ds, \qquad t \in \boldsymbol{R}^+,$$

and let (x^n, k^n) be a solution of the Skorokhod problem associated with y^n, $n \in N$. Then, for each $t \in \boldsymbol{R}^+$, $|x^n - x|_t \to 0$ and $|k^n - k|_t \to 0$, where (x, k) is a solution of the Skorokhod problem associated with y.

3. SDE's WITH REFLECTION ON AN ORTHANT

Let Z be an $(\mathscr{F}_t)$-adapted semimartingale. In this section we consider approximations of SDE's of the form (1.1). Let us recall that a pair (X, K) of $(\mathscr{F}_t)$-adapted processes is called a *strong solution* to (1.1) if (X, K) is a solution to the Skorokhod problem associated with the semimartingale

$$(3.1) \qquad Y_t = X_0 + \int_0^t \sigma(X_{s-})\, dZ_s, \qquad t \in \mathbf{R}^+.$$

LEMMA 3.1. *If σ is Lipschitz continuous, then there exists a unique strong solution to the SDE* (1.1).

Proof. We may and will assume that $\|Q\| < 1$. For $y \in \mathbf{D}(\mathbf{R}^+, \mathbf{R}^d)$ let $\Phi(y)$ denote a unique fixed point of the mapping (2.5) and let $\Gamma(y)\colon \mathbf{D}(\mathbf{R}^+, \mathbf{R}^d) \to \mathbf{D}(\mathbf{R}^+, \mathbf{R}^d)$ be a mapping defined by $\Gamma(y) = y + (I - Q')\Phi(y)$. Then $(\Gamma(y), \Phi(y))$ is a solution to the Skorokhod problem associated with y. Moreover, an easy computation shows that (X, K) is a solution to (1.1) if and only if Y defined by (3.1) is a solution to the following nonreflecting SDE with past-dependent coefficients

$$(3.2) \qquad Y_t = X_0 + \int_0^t \sigma(\Gamma(Y)_{s-})\, dZ_s, \qquad t \in \mathbf{R}^+.$$

Since the coordinates of $\sigma \circ \Gamma$ are functional Lipschitz in the sense of definition on p. 195 in [21], Chapter V, it follows from [21], Theorem 7, p. 197, that the SDE (3.2) has a unique strong solution, which completes the proof of the lemma. ∎

One can check that for each $n \in N$ the solutions $\hat{X}^n$, $\bar{X}^n$ of (1.2), (1.3) are given by the following recurrent formulas:

$$(3.3) \qquad \hat{X}^n_0 = \bar{X}^n_0 = X_0,$$

$$(3.4) \qquad \hat{X}^n_{(k+1)/n} = \Pi_Q(\hat{X}^n_{k/n}) + \sigma(\hat{X}^n_{k/n})(Z_{(k+1)/n} - Z_{k/n}),$$

$$(3.5) \qquad \bar{X}^n_{(k+1)/n} = \Pi_Q\big(\bar{X}^n_{k/n} + \sigma(\bar{X}^n_{k/n})(Z_{(k+1)/n} - Z_{k/n})\big),$$

$$(3.6) \qquad \hat{X}^n_t = \hat{X}^n_{k/n}, \qquad \bar{X}^n_t = \bar{X}^n_{k/n}, \qquad t \in [k/n, (k+1)/n),$$

where $k \in N \cup \{0\}$. In addition to (3.3)–(3.6) for $n \in N$, $k \in N \cup \{0\}$ set

$$\hat{K}^n_t = (I - Q')^{-1}\big(\hat{X}^n_t - X_0 - \int_0^t \sigma(\hat{X}^n_{s-})\, dZ^{\varrho^n}_s\big), \qquad t \in [k/n, (k+1)/n),$$

and denote by $\bar{K}^n$ the process appearing in (1.3).

THEOREM 3.2. *Assume that σ is Lipschitz continuous. Then*

$$(3.7) \qquad (\hat{X}^n, \hat{K}^n) \underset{\mathcal{P}}{\to} (X, K) \quad in\ (\mathbf{D}(\mathbf{R}^+, \mathbf{R}^{2d}), S),$$

and

$$(3.8) \qquad (\bar{X}^n, \bar{K}^n) \underset{\mathcal{P}}{\to} (X, K) \quad in\ (\mathbf{D}(\mathbf{R}^+, \mathbf{R}^{2d}), J_1),$$

where (X, K) is a unique strong solution to the SDE (1.1).

Proof. For $n \in N$ let $(\tilde{X}^n, \tilde{K}^n)$ denote the solution of the Skorokhod problem associated with

$$Y_t^{\varrho^n} = X_0 + \int_0^{\varrho_t^n} \sigma(X_{s-})\, dZ_s, \qquad t \in \boldsymbol{R}^+.$$

Then, by (2.10),

$$(3.9) \qquad (\tilde{X}^n, \tilde{K}^n) \to (X, K) \ \mathscr{P}\text{-a.s. in } \big(\boldsymbol{D}(\boldsymbol{R}^+, \boldsymbol{R}^{2d}), J_1\big).$$

Hence $(\tilde{X}^n, K^{\varrho^n}) \to (X, Z)$ $\mathscr{P}$-a.s. in $\big(\boldsymbol{D}(\boldsymbol{R}^+, \boldsymbol{R}^{2d}), J_1\big)$, which implies that

$$\varepsilon_t^n = \sup_{s \leqslant t} \Big|\int_0^t \sigma(\tilde{X}_{s-}^n)\, dZ_s^{\varrho^n} - \int_0^{\varrho_t^n} \sigma(X_{s-})\, dZ_s\Big| \underset{\mathscr{P}}{\to} 0, \qquad t \in \boldsymbol{R}^+.$$

On the other hand, by (2.4),

$$\sup_{s \leqslant t} |\bar{X}_s^n - \tilde{X}_s^n| \leqslant C\big\{\sup_{s \leqslant t} \Big|\int_0^t \sigma(\bar{X}_{s-}^n) - \sigma(\tilde{X}_{s-}^n)\, dZ_s^{\varrho^n}\Big| + \varepsilon_t^n\big\}, \qquad t \in \boldsymbol{R}^+.$$

Therefore, taking into account that σ is Lipschitz continuous, and using Lemma 5.3 (i) with $\alpha_n = 1$ yields $\sup_{s \leqslant t} |\bar{X}_s^n - \tilde{X}_s^n| \to_{\mathscr{P}} 0$, $t \in \boldsymbol{R}^+$, which gives (3.8), when combined with (3.9).

To prove (3.7) we first observe that

$$\big(\Pi_Q(\hat{X}_t^n), K_t^n\big) = \sum_{k;(k+1)/n \leqslant t} \big(\Pi_Q(\hat{X}_{(k+1)/n}^n) - \hat{X}_{(k+1)/n}^n\big), \qquad t \in \boldsymbol{R}^+,$$

is a solution of the Skorokhod problem associated with $X_0 + \int_0^t \sigma(\hat{X}_{s-}^n)\, dZ_s^{\varrho^n}$, $t \in \boldsymbol{R}^+$.

We next prove that

$$(3.10) \qquad \{\sup_{s \leqslant t} |\hat{X}_s^n - X_0|;\ n \in N\} \text{ is bounded in probability.}$$

To this end, set $Z_t^{\varrho^n, \varepsilon} = \sum_{s \leqslant t} \Delta Z_s^{\varrho^n} \mathbf{1}_{\{|\Delta Z_s^{\varrho^n}| > \varepsilon\}}(s)$ and assume that σ is Lipschitz continuous with a constant $L > 0$. Then, by (2.4) and (2.11),

$$\sup_{s \leqslant t} |\hat{X}_s^n - X_0| \leqslant C \sup_{s \leqslant t} \Big|\int_0^s \sigma(\hat{X}_{u-}^n)\, dZ_u^{\varrho^n}\Big| + C_Q \sup_{s \leqslant t} \int_0^s \|\sigma(\hat{X}_{u-}^n)\|\, d|Z^{\varrho^n, \varepsilon}|_u$$

$$+ C_Q\, \varepsilon \big(\|\sigma(X_0)\| + L \sup_{s \leqslant t} |\hat{X}_s^n - X_0|\big)$$

for every $\varepsilon > 0$. We can find $C', \varepsilon' > 0$ such that

$$\sup_{s \leqslant t} |\hat{X}_s^n - X_0| \leqslant C'\big\{1 + \sup_{s \leqslant t} \Big|\int_0^s \sigma(\hat{X}_{u-}^n)\, dZ_u^{\varrho^n}\Big| + \sup_{s \leqslant t} \int_0^s \|\sigma(\hat{X}_{u-}^n)\|\, d|Z^{\varrho^n, \varepsilon'}|_u\big\},$$

which together with Lemma 5.3 (ii) gives (3.10). Analysis similar to that in the proof of (2.11) shows that for $k \in N \cup \{0\}$, $n \in N$.

$$|\Pi_Q(\hat{X}^n_{(k+1)/n}) - \hat{X}^n_{(k+1)/n}| \leqslant C_Q |\sigma(\hat{X}^n_{k/n})(Z_{(k+1)/n} - Z_{k/n})|. \tag{3.11}$$

By (3.10) and (3.11), for every k, $n \in N$ there exists an $\mathscr{F}_{k/n}$-measurable random matrix C_{nk} such that

$$\Pi_Q(\hat{X}^n_{k/n}) - \hat{X}^n_{k/n} = C_{nk}(Z_{k/n} - Z_{(k-1)/n}),$$

and

$$\{\max_{k;k/n \leqslant t} \|C_{nk}\|;\ n \in N\} \text{ is bounded in probability.}$$

Furthermore, since σ is Lipschitz continuous, for k, $n \in N$ there exists an $\mathscr{F}_{k/n}$-measurable random matrix H_{nk} such that

$$\sigma(\Pi_Q(\hat{X}^n_{k/n})) - \sigma(\hat{X}^n_{k/n}) = H_{nk}(Z_{k/n} - Z_{(k-1)/n}), \tag{3.12}$$

and

$$\{\max_{k;k/n \leqslant t} \|H_{nk}\|;\ n \in N\} \text{ is bounded in probability.} \tag{3.13}$$

By (3.12), for every $\varepsilon > 0$

$$\delta^n_t = \sup_{s \leqslant t} \Big|\int_0^t \sigma(\hat{X}^n_{s-})\, dZ^{\varrho^n}_s - \int_0^t \sigma(\Pi_Q(\hat{X}^n_{s-}))\, dZ^{\varrho^n}_s\Big|$$

$$= \sup_{s \leqslant t} \Big| \sum_{k;(k+1)/n \leqslant t} H_{nk}\, \Delta Z^{\varrho^n}_{k/n}\, \Delta Z^{\varrho^n}_{(k+1)/n}\Big| \leqslant \delta^{n,1,\varepsilon}_t + \delta^{n,2,\varepsilon}_t,$$

where

$$\delta^{n,1,\varepsilon}_t = \sup_{s \leqslant t} \Big| \sum_{k \in K(n,\varepsilon)} H_{nk}\, \Delta Z^{\varrho^n}_{k/n}\, \Delta Z^{\varrho^n}_{(k+1)/n}\Big|,$$

$$\delta^{n,2,\varepsilon}_t = \sup_{s \leqslant t} \Big| \sum_{k \notin K(n,\varepsilon)} H_{nk}\, \Delta Z^{\varrho^n}_{k/n}\, \Delta Z^{\varrho^n}_{(k+1)/n}\Big|$$

and $K(n, \varepsilon) = \{k;\ (k+1)/n \leqslant t,\ |\Delta Z^{\varrho^n}_{k/n}| \leqslant \varepsilon\}$. Due to (3.13) and Remark 5.2 (ii), for every η one can find $\varepsilon' > 0$ such that

$$\sup_n \mathscr{P}(\delta^{n,1,\varepsilon'}_t > \eta) < \eta. \tag{3.14}$$

Since Z has trajectories in $D(R^+, R^d)$, it follows from (3.13) that $\delta^{n,2,\varepsilon}_t \to_{\mathscr{P}} 0$ for every $\varepsilon > 0$. Thus,

$$\delta^n_t \underset{\mathscr{P}}{\to} 0. \tag{3.15}$$

On the other hand, by (2.4),

$$\sup_{s\leqslant t}|\Pi_Q(\hat{X}^n_s)-\bar{X}^n_s| \leqslant C\sup_{s\leqslant t}\Big|\int_0^t \sigma(\hat{X}^n_{s-})-\sigma(\bar{X}^n_{s-})\,dZ^{\varrho^n}_s\Big|$$

$$\leqslant \Big\{C\sup_{s\leqslant t}\Big|\int_0^t \sigma\big(\Pi_Q(\hat{X}^n_{s-})\big)-\sigma(\bar{X}^n_{s-})\,dZ^{\varrho^n}_s\Big|+\delta^n_t\Big\}.$$

Hence, using Lipschitz continuity of σ, (3.15) and Gronwall's lemma we conclude that

$$\sup_{s\leqslant t}|\Pi_Q(\hat{X}^n_s)-\bar{X}^n_s| \underset{\mathscr{P}}{\to} 0, \qquad t\in \boldsymbol{R}^+.$$

Accordingly, $(\Pi_Q(\hat{X}^n), Z^{\varrho^n}) \to_{\mathscr{P}} (X, Z)$ in $(\boldsymbol{D}(\boldsymbol{R}^+, \boldsymbol{R}^{2d}), J_1)$ and, as a consequence,

$$(3.16)\qquad (\Pi_Q(\hat{X}^n), K^n) \underset{\mathscr{P}}{\to} (X, K) \quad \text{on } (\boldsymbol{D}(\boldsymbol{R}^+, \boldsymbol{R}^{2d}), J_1),$$

where $K^n_t = (I-Q')^{-1}(\Pi_Q(\hat{X}^n_t)-X_0-\int_0^t \sigma(\hat{X}^n_{s-})\,dZ^{\varrho^n}_s)$. Due to (3.11) and (3.16) we have $(\hat{X}^n_t, \hat{K}^n_t) \to_{\mathscr{P}} (X_t, K_t)$ for every $t\in\boldsymbol{R}^+$ such that $\mathscr{P}(\Delta Z_t = 0) = 1$ and $\{\sup_{s\leqslant t}(|\hat{X}^n_s|+|\hat{K}^n_s|)\}$ is bounded in probability. Finally, repeating the arguments from the proof of (2.9) in Theorem 2.4 shows that $\{(\hat{X}^n, \hat{K}^n)\}$ is S-tight, and (3.7) is proved. ∎

We now turn to the Euler–Peano scheme (1.4) which in some situations proved to be more convenient than the discrete penalization and the projection schemes (see e.g. [15]). First note that as in the case of the last two schemes the pair (X^n, K^n) can be constructed reccurently. Namely, for each $n\in N$ we set $X^n_0 = X_0$, and then if we have defined (X^n, K^n) for $t\in[0, k/n]$, then, for $t\in[k/n, (k+1)/n]$, (X^n, K^n) is a solution to the Skorokhod problem associated with $X^n_{k/n}+\sigma(X^n_{k/n})(Z_t-Z_{k/n})$, $t\in[k/n, (k+1)/n]$. In other words, (X^n, K^n) is a solution to the Skorokhod problem associated with Y^n, where

$$Y^n_t = X_0+\int_0^t \sigma(X^{n,\varrho^n}_{s-})\,dZ_s, \qquad t\in\boldsymbol{R}^+.$$

Observe that, similarly to (3.2), Y^n is a solution to the SDE

$$(3.17)\qquad Y^n_t = X_0+\int_0^t \sigma\big(\Gamma(Y^n_{\varrho^n_{s-}})\big)\,dZ_s, \qquad t\in\boldsymbol{R}^+.$$

THEOREM 3.3. *Assume that σ is Lipschitz continuous. Then for every $t\in\boldsymbol{R}^+$*

$$(3.18)\qquad \sup_{s\leqslant t}|X^n_t-X_t| \underset{\mathscr{P}}{\to} 0 \quad \textit{and} \quad \sup_{s\leqslant t}|K^n_t-X_t| \underset{\mathscr{P}}{\to} 0.$$

Moreover, if the sequence $\{\alpha_n\int_0^{\cdot}(Z^i_{s-}-Z^{\varrho^n,i}_{s-})\,dZ^j_s\}_{n\in N}$ *satisfies* (UT) (see the Appendix) *for* $i, j = 1, \ldots, d$ *for some sequence* $\{\alpha_n\}$ *of positive constants, then*

$$(3.19)\qquad \{\alpha_n(Y^n-Y)\}_{n\in N} \ \textit{satisfies}\ \text{(UT)}$$

and for every $t \in \boldsymbol{R}^+$

(3.20) $$\{\alpha_n \sup_{s \leqslant t} |X^n_s - X_s|\}_{n \in N} \text{ is bounded in probability.}$$

Proof. Since $\sigma \circ \Gamma$ is Lipschitz continuous for each j, $m = 1, \ldots, d$, the sequence $\{\sup_{t \leqslant q} |\sigma_{jm}(\Gamma(Y^n_{\varrho^n_{t-}}))|\}$ is bounded in probability and for every $t \in \boldsymbol{R}^+$ we have

(3.21) $$\begin{aligned} Y^{ni}_t - Y^i_t &= \sum_{j=1}^{d} \int_0^t \big(\sigma_{ij}(\Gamma(Y^n_{s-})) - \sigma_{ij}(\Gamma(Y_{s-}))\big)\, dZ^j_s \\ &\quad + \sum_{j=1}^{d} \int_0^t \big(\sigma_{ij}(\Gamma(Y^n_{\varrho^n_{s-}})) - \sigma_{ij}(\Gamma(Y_{s-}))\big)\, dZ^j_s \\ &= \sum_{j=1}^{d} \int_0^t U^{nij}_s (Y^{nj}_{s-} - Y^j_{s-})\, dZ^j_s \\ &\quad + \sum_{j=1}^{d} \sum_{m=1}^{d} \int_0^t V^{nij}_s \sigma_{jm}(\Gamma(Y^n)_{\varrho^n_{s-}})(Z^m_{s-} - Z^{m,\varrho^n}_{s-})\, dZ^j_s, \end{aligned}$$

where U^{nij} and V^{nij} are predictable processes such that $\sup_{t \leqslant q} |U^{nij}_t| \leqslant C$ and $\sup_{t \leqslant q} |V^{nij}_t| \leqslant C$ for some $C > 0$. By the same method as in the proof of Lemma 3.2 in [14],

$$\sup_{s \leqslant t} \Big|\int_0^s V^{nij}_s \sigma_{jm}(\Gamma(Y^n)_{\varrho^n_{s-}})(Z^m_{u-} - Z^{m,\varrho^n}_{u-})\, dZ^j_u\Big| \underset{\mathscr{P}}{\to} 0, \qquad t \in \boldsymbol{R}^+,$$

for m, $j = 1, \ldots, d$. By (3.21) and Lemma 5.3,

$$\sup_{s \leqslant t} |Y^n_s - Y_s| \underset{\mathscr{P}}{\to} 0, \qquad t \in \boldsymbol{R}^+,$$

and hence (3.18) follows. In order to prove (3.19) it is sufficient to multiply (3.21) by α_n and use the arguments from the proof of Theorem 3.6 in [26]. Since (3.19) implies that $\{\alpha_n \sup_{s \leqslant t} |Y^n_s - Y_s|\}$ is bounded in probability and Γ is Lipschitz continuous, (3.20) follows. ■

THEOREM 3.4. *Assume that* σ *is Lipschitz continuous and that, for some* $1 \leqslant k \leqslant d-1$, $\sigma_{ij}(x) = \sigma_{ij}$ *for all* $x \in (\boldsymbol{R}^+)^d$ *and* i, j *such that* $\min(i, j) \leqslant k$. *If the components* $Z^{k+1}, \ldots, Z^d$ *are processes with locally bounded variation, then*

(3.22) $$|X^n - X|_t \underset{\mathscr{P}}{\to} 0, \qquad t \in \boldsymbol{R}^+.$$

Moreover, if for every $j = 1, \ldots, d$ *and* $i = k+1, \ldots, d$ *the sequence* $\{\alpha_n \int_0^t |Z^j_{s-} - Z^{\varrho^n, j}_{s-}|\, d|Z^i|_s\}$ *is bounded in probability for some sequence* $\{\alpha_n\}$ *of positive constants, then*

(3.23) $$\{\alpha_n |Y^n - Y|_t\} \text{ is bounded in probability, } t \in \boldsymbol{R}^+,$$

and

(3.24) $\{\alpha_n |X^n - X|_t\}$ *is bounded in probability*, $t \in \boldsymbol{R}^+$.

Proof. By (3.21), for $i = k+1, \ldots, d$ we have

$$|Y^{ni} - Y^i|_t \leqslant \sum_{j=k+1}^{d} \int_0^t |U_s^{nij}| \, |Y^{nj} - Y^j|_{s-} \, d|Z^j|_s$$

$$+ \sum_{j=k+1}^{d} \sum_{m=1}^{d} \int_0^t |V_s^{nij}| \, |\sigma_{jm}(\Gamma(Y^n)_{\varrho_{s-}^n})| \, |Z_{s-}^m - Z_{s-}^{m,\varrho^n}| \, d|Z^j|_s.$$

At the same time for $j = k+1, \ldots, d$

$$\int_0^t |V_s^{nij}| \, |\sigma_{jm}(\Gamma(Y^n)_{\varrho_{s-}^n})| \, |Z_{s-}^m - Z_{s-}^{m,\varrho^n}| \, d|Z^j|_s \underset{\mathscr{P}}{\rightarrow} 0, \qquad t \in \boldsymbol{R}^+,$$

so (3.22)–(3.24) follow from Lemma 5.3. ∎

4. DIFFUSIONS REFLECTING ON AN ORTHANT

In this section we consider the solutions of the SDE (1.5). We will assume that

(4.1) $$\|\sigma(x) - \sigma(y)\| + |b(x) - b(y)| \leqslant L|x-y|, \qquad x, y \in \boldsymbol{R}^d.$$

By Lemma 3.1, under (4.1) there exists a unique strong solution (X, K) to the SDE (1.5). Moreover, in view of (2.4) and Gronwall's lemma, $E\sup_{s \leqslant t}|X_s|^{2p} < +\infty$ for every $p \in \boldsymbol{N}$ and $t \in \boldsymbol{R}^+$.

In the present situation the formulas for $\hat{X}^n$ and $\bar{X}^n$ take the forms (3.3)–(3.6) with $\sigma(\hat{X}_{k/n}^n)(Z_{(k+1)/n} - Z_{k/n})$ and $\sigma(\bar{X}_{k/n}^n)(Z_{(k+1)/n} - Z_{k/n})$ replaced by

$$b(\hat{X}_{k/n}^n)n^{-1} + \sigma(\hat{X}_{k/n}^n)(W_{(k+1)/n} - W_{k/n}) \quad \text{and} \quad b(\bar{X}_{k/n}^n)n^{-1} + \sigma(\bar{X}_{k/n}^n)(W_{(k+1)/n} - W_{k/n}).$$

Note also that now $(\bar{X}^n, \bar{K}^n)$ is the solution of the Skorokhod problem associated with

$$X_0 + \int_0^t \sigma(\bar{X}_{s-}^n) \, dW_s^{\varrho^n} + \int_0^t b(\bar{X}_{s-}^n) \, d\varrho_s^n, \qquad t \in \boldsymbol{R}^+,$$

and

$$(\Pi_Q(\hat{X}_t^n), K_t^n) = \sum_{k;(k+1)/n \leqslant t} \hat{X}_{(k+1)/n}^n - \Pi_Q(\hat{X}_{(k+1)/n}^n), \qquad t \in \boldsymbol{R}^+,$$

is a solution to the Skorokhod problem associated with

$$X_0 + \int_0^t \sigma(\hat{X}_{s-}^n) \, dW_s^{\varrho^n} + \int_0^t b(\hat{X}_{s-}^n) \, d\varrho_s^n, \qquad t \in \boldsymbol{R}^+.$$

As in the case of (X, K), using (2.4), (4.1) and Gronwall's lemma yields

$$\sup_{n\in N} E\sup_{s\leqslant t}|\bar{X}_s^n|^{2p} < +\infty, \qquad \sup_{n\in N} E\sup_{s\leqslant t}|\hat{X}_s^n|^{2p} < +\infty \tag{4.2}$$

for every $p\in N$ and $t\in \boldsymbol{R}^+$.

THEOREM 4.1. *Assume that* (4.1) *holds and let* (X, K) *be a strong solution to the SDE* (1.5). *Then for every* $p\in N$

$$E\sup_{s\leqslant t}|\bar{X}_s^n - X_s|^{2p} = \mathcal{O}\left(((\ln n)/n)^p\right), \qquad t\in \boldsymbol{R}^+, \tag{4.3}$$

and

$$E\sup_{s\leqslant t}|\hat{X}_s^n - X_s|^{2p} = \mathcal{O}\left(((\ln n)/n)^p\right), \qquad t\in \boldsymbol{R}^+. \tag{4.4}$$

Proof. Let $(\tilde{X}^n, \tilde{K}^n)$ denote a solution of the Skorokhod problem associated with $X_0+\int_0^t \sigma(\bar{X}_{s-}^n)\,dW_s+\int_0^t b(\bar{X}_{s-}^n)\,ds$, $t\in \boldsymbol{R}^+$. By (2.4) there is $C>0$ such that

$$E\sup_{s\leqslant t}|\bar{X}_s^n-\tilde{X}_s^n|^{2p} \leqslant C^{2p} E\sup_{s\leqslant t}\Big|\int_{\varrho_s^n}^{s}\sigma(\bar{X}_{u-}^n)\,du+\int_{\varrho_s^n}^{s} b(\bar{X}_{u-}^n)\,du\Big|^{2p}$$

$$\leqslant \mathrm{Const}\, E\{\sup_{s\leqslant t}\|\sigma(\bar{X}_{s-}^n)\|^{2p}|W_s-W_{\varrho_s^n}|^{2p}+\sup_{s\leqslant t}|b(\bar{X}_{s-}^n)|^{2p}|s-\varrho_s^n|^{2p}\}$$

$$\leqslant \mathrm{Const}\, \{(E\sup_{s\leqslant t}\|\sigma(\bar{X}_{s-}^n)\|^{4p})^{1/2}(E\omega_W(n^{-1}, t)^{4p})^{1/2}+(n^{-1})^{2p}E\sup_{s\leqslant t}|b(\bar{X}_{s-}^n)|^{2p}\}.$$

Therefore, using (4.2) and the fact that

$$E\omega_W(n^{-1}, t)^{2p} = \mathcal{O}\left(((\ln n)/n)^p\right), \qquad t\in \boldsymbol{R}^+, \tag{4.5}$$

which can be easily deduced from Lemma A4 in [26], we obtain

$$E\sup_{s\leqslant t}|\bar{X}_s^n-\tilde{X}_s^n|^{2p} \leqslant \mathrm{Const}\left((\ln n)/n\right)^p, \qquad t\in \boldsymbol{R}^+. \tag{4.6}$$

Similarly,

$$\begin{aligned} &E\sup_{s\leqslant t}|\tilde{X}_s^n-X_s|^{2p} \\ &\leqslant \mathrm{Const}\, E\{\sup_{s\leqslant t}\int_0^s\|\sigma(\bar{X}_{u-}^n)-\sigma(X_u)\|^{2p}\,du+\sup_{s\leqslant t}\int_0^s|b(\bar{X}_{u-}^n)-b(X_u)|^{2p}\,du\} \\ &\leqslant \mathrm{Const}\int_0^t E\sup_{s\leqslant u}|\bar{X}_{s-}^n-X_s|^{2p}\,du \leqslant \mathrm{Const}\int_0^t E\sup_{s\leqslant u}|\bar{X}_s^n-X_s|^{2p}\,du. \end{aligned} \tag{4.7}$$

Combining (4.6) with (4.7) gives

$$E\sup_{s\leqslant t}|\bar{X}^n_s - X_s|^{2p} \leqslant \mathrm{Const}\,\{((\ln n)/n)^p + \int_0^t E\sup_{s\leqslant u}|\bar{X}^n_s - X_s|^{2p}\,du\},$$

and (4.3) follows by Gronwall's lemma. By (2.6), for every $k\in N\cup\{0\}$ we obtain

$$|\Pi_Q(\hat{X}^n_{(k+1)/n}) - \hat{X}^n_{(k+1)/n}| \leqslant C_Q\,\{|\sigma(\hat{X}^n_{k/n})(W_{(k+1)/n} - W_{k/n})| + |b(\hat{X}^n_{k/n})|\,n^{-1}\}$$

and, consequently,

$$\begin{aligned} (4.8)\qquad & E\sup_{s\leqslant t}|\Pi_Q(\hat{X}^n_s) - \hat{X}^n_s|^{2p} \\ & \leqslant \mathrm{Const}\,\{(E\sup_{s\leqslant t}|\hat{X}^n_s|^{4p}+1)^{1/2}\big((E\omega_W(n^{-1}, t)^{4p})^{1/2} + (n^{-1})^{2p}\big)\} \\ & \leqslant \mathrm{Const}\,((\ln n)/n)^p. \end{aligned}$$

Therefore, in view of (4.3) to complete the proof of (4.4) it is sufficient to show that for every $p\in N$

$$(4.9)\qquad E\sup_{s\leqslant t}|\bar{X}^n_s - \Pi_Q(\hat{X}^n_s)|^{2p} \leqslant \mathcal{O}\big(((\ln n)/n)^p\big), \qquad t\in \boldsymbol{R}^+.$$

To prove this, we first observe that, by (2.4) and (4.1),

$$\begin{aligned} & E\sup_{s\leqslant t}|\bar{X}^n_s - \Pi_Q(X^n_s)|^{2p} \\ & \leqslant C^{2p}E\,\{[\int_0^{\cdot}(\sigma(\bar{X}^n_{s-}) - \sigma(\hat{X}^n_{s-}))\,dW^{\varrho^n}_s]^p_t + [\int_0^{\cdot}(b(\bar{X}^n_{s-}) - b(\hat{X}^n_{s-}))\,d\varrho^n_s]^p_t\} \\ & \leqslant \mathrm{Const}\,E\,\{\sup_{s\leqslant t}|\int_0^s(\sigma(\bar{X}^n_{u-}) - \sigma(\hat{X}^n_{u-}))\,dW^{\varrho^n}_u|^{2p} + \int_0^t\sup_{u\leqslant s}|\bar{X}^n_{u-} - \hat{X}^n_{u-}|^{2p}\,ds\} \\ & = \mathrm{Const}\int_0^t E\sup_{u\leqslant s}|\bar{X}^n_u - \hat{X}^n_u|^{2p}\,ds. \end{aligned}$$

Hence, by (4.8),

$$E\sup_{s\leqslant t}|\bar{X}^n_s - \Pi_Q(\hat{X}^n_s)|^{2p} \leqslant \mathrm{Const}\left\{\left(\frac{\ln n}{n}\right)^p + \int_0^t E\sup_{u\leqslant s}|\bar{X}^n_s - \Pi(\hat{X}^n_u)|^{2p}\,ds\right\},$$

which leads to (4.9) by Gronwall's lemma. ∎

For the Euler–Peano scheme $(X^n_0, K^n_0) = (X_0, 0)$ and for $t\in[k/n, (k+1)/n]$ (X^n, K^n) is a solution to the Skorokhod problem associated with

$$X^n_{k/n} + \sigma(X^n_{k/n})(W_t - W_{k/n}) + b(X^n_{k/n})(t - k/n).$$

THEOREM 4.2. *Assume that* (4.1) *holds and let* (X, K) *be a strong solution to the SDE* (1.5). *Then for every* $p\in N$

$$(4.10)\qquad E\sup_{s\leqslant t}|X^n_s - X_s|^{2p} = \mathcal{O}((n^{-1})^p), \qquad t\in\boldsymbol{R}^+.$$

If additionally $\sigma(x) = \sigma$ *for all* $x \in (\mathbf{R}^+)^d$, *then*

(4.11) $$E|X^n - X|_t^p = \mathcal{O}\big((n^{-1})^{p/2}\big), \qquad t \in \mathbf{R}^+.$$

Proof. Since (X^n, K^n) is a solution to the Skorokhod problem associated with Y^n given by

$$Y_t^n = X_0 + \int_0^t \sigma(X_{s-}^{n,\varrho^n})\,dW_s + \int_0^t b(X_{s-}^{n,\varrho^n})\,ds, \qquad t \in \mathbf{R}^+,$$

from (2.4) we conclude that

$$E\sup_{s\leqslant t}|X_s^n - X_s|^{2p} \leqslant \mathrm{Const}\Big\{\int_0^t E\sup_{u\leqslant s}|X_u^n - X_u|^{2p}\,ds$$
$$+E\sup_{s\leqslant t}\int_0^s \big(\|\sigma(X_{u-}^{n,\varrho^n})\|^{2p}|W_{u-}^{n,\varrho^n} - W_u|^{2p} + |b(X_{u-}^{n,\varrho^n})|^{2p}(u-\varrho_u^n)\big)\,du\Big\}.$$

By (4.1) and Hölder's inequality the last term in bounded by

$$\mathrm{Const}\Big\{\big(1+E\sup_{s\leqslant t}|X_{s-}^{n,\varrho^n}|^{4p}\big)^{1/2}\big(\big(E\omega_W(n^{-1},t)^{4p}\big)^{1/2} + (n^{-1})^{2p}\big)\Big\}.$$

Therefore (4.10) follows from (4.5) and Gronwall's lemma. The proof of (4.11) is similar, the only difference being in the use of (2.16) instead of (2.4). ∎

From Theorem 4.2 and the Markov inequality it follows immediately that for each $t \in \mathbf{R}^+$

(4.12) $$\{\sqrt{n}\sup_{s\leqslant t}|X_s^n - X_s|\}_{n\in N} \text{ is bounded in probability}$$

and

(4.13) $$\{\sqrt{n}|X^n - X|_t\}_{n\in N} \text{ is bounded in probability,}$$

provided that $\sigma(x) = \sigma$. Let us remark, however, that (4.12) and (4.13) may be also deduced from Theorem 3.4, because for $i, j = 1, \ldots, d$ the sequence $\{\sqrt{n}\int_0^{\bullet}(W_{s-}^i - W_{s-}^{\varrho^n,i})\,dW_s^j\}_{n\in N}$ satisfies (UT) (see, e.g., [25], p. 42) and the sequence $\{\sqrt{n}\int_0^t|W_{s-}^i - W_{s-}^{\varrho^n,i}|\,ds\}_{n\in N}$ is bounded in probability.

5. APPENDIX

Let Z be an $(\mathcal{F}_t)$-adapted semimartingale of the form

(5.1) $$Z_t = Z_0 + J_t + M_t + V_t, \qquad t \in \mathbf{R}^+,$$

where $J_t = \sum_{s\leqslant t}\Delta Z_s \mathbf{1}_{\{|\Delta Z_s|>1\}}$, $t \in \mathbf{R}^+$, M is a locally square-integrable martingale with $M_0 = 0$ and V is a predictable process with locally bounded variation and $V_0 = 0$.

LEMMA 5.1. *Assume that X and H are $(\mathscr{F}_t)$-adapted processes.*

(i) *If for every $t \in \mathbf{R}^+$*

$$\sup_{s \leqslant t} |X_s| \leqslant \sup_{s \leqslant t} \Big|\int_0^s X_{u-}\, dZ_u\Big| + \sup_{s \leqslant t} |H_s|,$$

then there is a universal constant C (depending only on d) such that for every $\varepsilon, \eta, \delta > 0$ and for every $(\mathscr{F}_t)$-stopping time σ

$$\mathscr{P}(\sup_{t \leqslant \sigma} |X_t| \geqslant \varepsilon) \leqslant \mathscr{P}(\sup_{t \leqslant \sigma} |H_t| \geqslant \eta) + \mathscr{P}(B_\sigma \geqslant \delta) + \varepsilon^{-2} \eta^2 \exp\{\delta(1+\delta)C\},$$

where $B_t = [M]_t + \langle M \rangle_t + |J|_t + |V|_t$, $t \in \mathbf{R}^+$.

(ii) *If Z is a process with locally bounded variation and for every $t \in \mathbf{R}^+$*

$$|X|_t \leqslant \Big|\int_0^{\cdot} X_{s-}\, dZ_s\Big|_t + \sup_{s \leqslant t} |H_s|,$$

then for every $\varepsilon, \eta, \delta > 0$ and for every $(\mathscr{F}_t)$-stopping time σ

$$\mathscr{P}(|X|_\sigma \geqslant \varepsilon) \leqslant \mathscr{P}(\sup_{t \leqslant \sigma} |H_t| \geqslant \eta) + \mathscr{P}(|Z|_\sigma \geqslant \delta) + \varepsilon^{-1} \eta \exp\{\delta\}.$$

Proof. Define $\tau = \inf\{t;\ B_t \geqslant \delta \text{ or } \sup_{s \leqslant t} |H_s| \geqslant \eta\}$. Then

$$\mathscr{P}(\sup_{t \leqslant \sigma} |X_t| \geqslant \varepsilon) \leqslant \mathscr{P}(\sup_{t \leqslant \sigma} |H_t| \geqslant \eta) + \mathscr{P}(B_\sigma \geqslant \delta) + \mathscr{P}(\sup_{t \leqslant \sigma} |X_t| \geqslant \varepsilon,\ \tau > \sigma).$$

By Chebyshev's inequality the last term on the right-hand side is bounded by $\varepsilon^{-2} E \sup_{t \leqslant \sigma} |X_t^{\tau-}|^2$. On the other hand, for every $(\mathscr{F}_t)$-stopping time γ

$$\sup_{t<\gamma} |X_t^{\tau-}|^2 \leqslant 2\{\sup_{t<\gamma} |H_t^{\tau-}|^2 + \sup_{t<\gamma\wedge\tau} \Big|\int_0^t X_{s-}\, dZ_s\Big|^2\}$$

$$\leqslant 2\eta^2 + 6C_1(d)\{\delta \sup_{t<\gamma\wedge\tau} \int_0^t |X_{s-}^{\tau}|^2\, d|J|_s + \delta \sup_{t<\gamma\wedge\tau} \int_0^t |X_{s-}|^2\, d|V|_s\}$$

$$+6 \sup_{t<\gamma\wedge\tau} \Big|\int_0^t X_{s-} dM_s\Big|^2$$

for some $C_1(d)$ depending only on d, and, by Doob's type inequality proved in [18],

$$E \sup_{t<\gamma\wedge\tau} \Big|\int_0^t X_{s-}\, dM_s\Big|^2 \leqslant C_2(d) E \int_0^{(\gamma\wedge\tau)-} |X_{s-}|^2\, d([M]_s + \langle M \rangle_s).$$

Hence $E \sup_{t<\gamma} |X^{\tau-}|^2 \leqslant 2\eta^2 + C_3(d)(1+\delta) E \int_0^{\gamma^-} \sup_{u \leqslant s} |X_{u-}^{\tau-}|^2\, dB_s^{\tau-}$. Therefore, applying Lemma 3 of [23] gives

$$E \sup_t |X_t^{\tau-}|^2 \leqslant 2\eta^2 \exp\{\delta(1+\delta) C_3(d)\},$$

and the proof of (i) is complete. The proof of (ii) is similar, so we omit it. ∎

For $n \in N$ let Z^n be an $(\mathscr{F}_t^n)$-adapted semimartingale. Following Stricker [27] we say that $\{Z^n\}$ *satisfies* (UT) if for every $q \in R^+$ the family of random variables

$$\Big\{ \int_{[0,q]} U_s^n \, dZ_s^n;\ n \in N,\ U^n \in U_q^n \Big\} \text{ is bounded in probability,}$$

where U_q^n is the class of discrete predictable processes of the form

$$U_s^n = U_0^n + \sum_{i=0}^{k} U_i^n \mathbf{1}_{\{t_i < s \leqslant t_{i+1}\}},$$

where $0 = t_0 < t_1 < \ldots < t_k = q$, U_i^n is $\mathscr{F}_{t_i}^n$-measurable and $|U_i^n| \leqslant 1$ for every $i \in N \cup \{0\}$, $n \in N$, $k \in N$. Some sufficient conditions for (UT) and some examples of its applications can be found in Jakubowski et al. [11], Kurtz and Protter [13], [14], Mémin and Słomiński [17] and Słomiński [23], [25].

Remark 5.2. (i) Let Z be an $(\mathscr{F}_t)$-adapted semimartingale and let Z^{ϱ^n} be a discretization of Z adapted to the discrete filtration $\mathscr{F}^{\varrho^n}$, $n \in N$. Then by the famous theorem of Bichteler, Dellacherie and Mokobodski (see e.g. [3], Theorem 2.5) it follows that $\{Z^{\varrho^n}\}$ satisfies (UT).

(ii) In [17] it was proved that a sequence of semimartingales $\{Z^n\}$ of the form $Z_t^n = Z_0^n + J_t^n + M_t^n + V_t^n$, $t \in R^+$, satisfies (UT) if and only if $\sup_n |Z_0^n| < +\infty$ and the families $\{|J^n|_t;\ n \in N\}$, $\{[M^n]_t;\ n \in N\}$ and $\{|V|_t^n;\ n \in N\}$ are bounded in probability, $t \in R^+$. From this one can deduce in particular that if $\{Z^n\}$ is a sequence of semimartingales satisfying (UT) and $\{H^n\}$ is a sequence of predictable processes such that $\{\sup_{s \leqslant t} |H_s^n|;\ n \in N\}$ is bounded in probability, then the sequence of stochastic integrals $\{\int_0^{\cdot} H_s^n \, dZ_s^n\}$ satisfies (UT) as well.

LEMMA 5.3. *Let σ be a Lipschitz continuous function and for $n \in N$ let Z^n, X^n, $\tilde{X}^n$, H^n be $(\mathscr{F}_t^n)$-adapted processes such that*

$$\alpha_n \sup_{s \leqslant t} |X_s^n - \tilde{X}_s^n| \leqslant C \sup_{s \leqslant t} \Big| \int_0^s \alpha_n \big(\sigma(X_{s-}^n) - \sigma(\tilde{X}_{s-}^n)\big) \, dZ_s^n \Big| + \alpha_n \sup_{s \leqslant t} |H_s^n|, \qquad t \in R^+$$

$$(\textit{respectively, } \ \alpha_n |X^n - \tilde{X}^n|_t \leqslant C \Big| \int_0^{\cdot} \alpha_n \big(f(X_{s-}^n) - f(\tilde{X}_{s-}^n)\big) \, dZ_s^n \Big|_t + \alpha_n \sup_{s \leqslant t} |H_s^n|, \quad t \in R^+)$$

for some sequence $\{\alpha_n\} \subset R^+$. If $\{Z^n\}$ satisfies (UT) *(respectively, $\{|Z^n|_t;\ n \in N\}$ is bounded in probability for $t \in R^+$), then for every $t \in R^+$ the following implications hold true:*

(i) *if $\alpha_n \sup_{s \leqslant t} |H_s^n| \to_{\mathscr{P}} 0$, then also*

$$\alpha_n \sup_{s \leqslant t} |X_s^n - \tilde{X}_s^n| \underset{\mathscr{P}}{\to} 0 \qquad (\textit{respectively, } \ \alpha_n |X^n - \tilde{X}^n|_t \underset{\mathscr{P}}{\to} 0);$$

(ii) *if $\{\alpha_n \sup_{s \leqslant t} |H_s^n|;\ n \in N\}$ is bounded in probability, then also*

$$\{\alpha_n \sup_{s \leqslant t} |X_s^n - \tilde{X}_s^n|;\ n \in N\} \textit{ is bounded in probability}$$

(respectively, $\{\alpha_n |X^n - \tilde{X}^n|_t;\ n \in N\}$ is bounded in probability).

Proof. By Remark 5.2 (ii), $\{Z^n\}$ satisfies (UT) if and only if

$$\{B_t^n = [M^n]_t + \langle M^n \rangle_t + |J^n|_t + |V^n|_t\} \text{ is bounded in probability,}$$

so the required results follow from Lemma 5.1. ∎

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